

ON AUTOMORPHISMS OF LIE ALGEBRAS OF CLASSICAL TYPE

BY

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In two recent papers⁽¹⁾ ([12] and [13], the latter hereafter referred to as (C)), W. H. Mills and the author have developed a structure theory for certain Lie algebras over arbitrary fields of characteristic different from 2 and 3. This class of algebras is defined by axioms in (C), and its members will be referred to as Lie algebras of classical type. They are closely related to semi-simple Lie algebras over algebraically closed fields of characteristic zero, not only in that such algebras are of classical type, but also in that nearly every Lie algebra of classical type may be obtained from such an algebra of characteristic zero by reducing its integral structure constants modulo a prime and performing a suitable extension of the prime field [2; 3; 4; 15].

There are several reasons for seeking information about the automorphisms of these Lie algebras. In case the base field \mathfrak{F} is finite, recent investigations by Chevalley [2] have revealed new simple finite groups closely related to the automorphism groups of the "exceptional" simple Lie algebras of classical type over \mathfrak{F} . It would be of interest to obtain more precise information as to the relations between these groups. If one is to make an analysis of all normal simple Lie algebras over arbitrary fields of characteristic different from 2 or 3 which become simple Lie algebras of classical type when one passes to the algebraic closure (e.g., normal simple algebras with nondegenerate Killing forms), the techniques used in the past require a knowledge of the automorphisms in the algebraically closed case [9; 10; 11; 17]. Since the groups which occur as automorphism groups include several of the classical groups our viewpoint represents an alternate, if perhaps devious, method for getting information about the classical groups. In the case of the automorphism groups of the exceptional algebras, the scarcity of knowledge on these linear groups lends interest to any pertinent results. It is proposed to discuss the meaning of our results for the various linear groups in a later paper.

The techniques utilized are basically of two kinds. First, the analogy of simple Lie algebras of classical type with simple Lie algebras over algebraically closed fields of characteristic zero is refined so as to verify a number of results on roots and the Weyl group in the general case. This procedure, which is carried out in II, was developed for Lie algebras with nondegenerate Killing forms by Curtis [3; 4], and our techniques are basically the same

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⁽¹⁾ Numbers in brackets refer to the bibliography at the end of the paper.

as his. Secondly, in III and IV the transcendental techniques of Gantmacher with exponentials [7] are replaced by the use of iterated exponentials which are defined over all fields of characteristic different from 2 and 3. These mappings have been discussed from a somewhat different viewpoint by Chevalley [2]. Considerable similarity with his techniques will be noted. The results on the situation of the group of invariant [15] automorphisms in the full automorphism group in the algebraically closed case are stated in V.

I. Automorphisms and standard Cartan subalgebras. Let \mathfrak{L} be a Lie algebra of classical type, in the sense of (C). Let σ be an automorphism of \mathfrak{L} and \mathfrak{H} an abelian Cartan subalgebra relative to which (iii)–(v) of (C) are satisfied (such a Cartan subalgebra will in the sequel be called a *standard* Cartan subalgebra). Then $\mathfrak{H}\sigma$ is again an abelian Cartan subalgebra of \mathfrak{L} , and if $\mathfrak{L} = \mathfrak{H} + \sum_{\alpha} \mathfrak{L}_{\alpha}$ is the Cartan decomposition relative to \mathfrak{H} , we have $\mathfrak{L} = \mathfrak{H}\sigma + \sum_{\alpha} \mathfrak{L}_{\alpha}\sigma$, again a vector space direct sum. Moreover, if $e_{\alpha} \in \mathfrak{L}_{\alpha}$, then $[e_{\alpha}\sigma, h\sigma] = [e_{\alpha}h]\sigma = \alpha(h)e_{\alpha}\sigma$ for all $h \in \mathfrak{H}$; that is, $\mathfrak{L}_{\alpha}\sigma$ is a root-space relative to $\mathfrak{H}\sigma$, and the corresponding root is α^{σ} : $\alpha^{\sigma}(h\sigma) = \alpha(h)$. Thus $\mathfrak{L} = \mathfrak{H}\sigma + \sum_{\alpha} \mathfrak{L}_{\alpha}\sigma$ is the Cartan decomposition of \mathfrak{L} relative to $\mathfrak{H}\sigma$. Also $[\mathfrak{L}_{\alpha}\sigma, \mathfrak{L}_{-\alpha}\sigma] = [\mathfrak{L}_{\alpha}, \mathfrak{L}_{-\alpha}]\sigma$ is one-dimensional, and if α^{σ} is a nonzero root relative to $\mathfrak{H}\sigma$ and $\beta \in (\mathfrak{H}\sigma)^*$, then $\beta = \gamma^{\sigma}$, where $\gamma \in \mathfrak{H}^*$ and $\beta(h\sigma) = \gamma(h)$ for all $h \in \mathfrak{H}$. Since $\gamma \rightarrow \gamma^{\sigma}$ is evidently a linear mapping of \mathfrak{H}^* onto $(\mathfrak{H}\sigma)^*$, we see that if all the forms $\beta + m\alpha^{\sigma} = (\gamma + m\alpha)^{\sigma}$ (m an integer) were roots relative to $\mathfrak{H}\sigma$, then all the forms $\gamma + m\alpha$ would be roots relative to \mathfrak{H} , contrary to (v). Thus (iii)–(v) of (C) are satisfied by \mathfrak{L} relative to $\mathfrak{H}\sigma$; $\mathfrak{H}\sigma$ is a standard Cartan subalgebra.

Now $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_s$, where the \mathfrak{L}_i are nonabelian simple ideals satisfying the same axioms as \mathfrak{L} does ((C), Theorem 4). Then it is well known that each $\mathfrak{L}_i\sigma$ is an \mathfrak{L}_j , i.e., σ permutes the \mathfrak{L}_i . Of course, $\mathfrak{L}_i\sigma = \mathfrak{L}_j$ is only possible if \mathfrak{L}_i and \mathfrak{L}_j are isomorphic simple algebras. If we collect all the isomorphic \mathfrak{L}_i in the decomposition $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_s$ into blocks, then σ effects a permutation in each block. We cannot hope to achieve this permutation through operations generated by the Lie product in the algebra, since these will always map each \mathfrak{L}_i into itself. However, after performing one of the finite set of all possible permutations of the \mathfrak{L}_i in each block, we can replace σ by an automorphism which maps each \mathfrak{L}_i into itself. In this sense the problem of describing all automorphisms of \mathfrak{L} is reduced to the case where \mathfrak{L} is simple.

We therefore assume \mathfrak{L} is simple, we let \mathfrak{H} be as above, and we let $\alpha_1, \dots, \alpha_r$ be a *fundamental system* of roots relative to \mathfrak{H} . That is, $\alpha_1, \dots, \alpha_r$ span \mathfrak{H}^* , $\alpha_i - \alpha_j$ is not a root if $i \neq j$, and the $r \times r$ matrix (A_{ij}) of rational integers ((C), (xiv) and §7) is nonsingular; moreover, (A_{ij}) is *irreducible*: for every $i \neq j$, there are indices $i_0 = i, i_1, i_2, \dots, i_k = j$ such that $A_{i_\nu, i_{\nu+1}} \neq 0$, $0 \leq \nu \leq k-1$. Finally, there is a unique root β relative to \mathfrak{H} such that no $\beta - \alpha_i$ is a root, $1 \leq i \leq r$ ((C), §10). Since the roots relative to $\mathfrak{H}\sigma$ span $(\mathfrak{H}\sigma)^*$ and since $\alpha \rightarrow \alpha^{\sigma}$ is linear from \mathfrak{H}^* onto $(\mathfrak{H}\sigma)^*$ and maps the set of roots relative to \mathfrak{H} onto that relative to $\mathfrak{H}\sigma$, it follows that $\alpha_1^{\sigma}, \dots, \alpha_r^{\sigma}$ span $(\mathfrak{H}\sigma)^*$.

If $\alpha_i^\sigma - \alpha_j^\sigma$ were a root for some $i \neq j$, then so would be $(\alpha_i^\sigma - \alpha_j^\sigma)^{\sigma^{-1}} = \alpha_i - \alpha_j$, a contradiction. The integer $A_{ij} = A_{\alpha_i, \alpha_j}$ is determined by the conditions that $A_{ii} = 2$, and if $i \neq j$, $-A_{ij} = m - 1$, where m is the first non-negative integer such that $\alpha_i + m\alpha_j$ is not a root. Now $A_{\alpha_i^\sigma, \alpha_i^\sigma} = 2$ by definition, and $-A_{\alpha_i^\sigma, \alpha_j^\sigma} = t - 1$, where t is the first non-negative integer such that $\alpha_i^\sigma + t\alpha_j^\sigma$ is not a root. But $\alpha_i + m\alpha_j$ is a root if and only if $(\alpha_i + m\alpha_j)\sigma = \alpha_i^\sigma + m\alpha_j^\sigma$ is a root relative to $\mathfrak{H}\sigma$. Thus $m = t$ and $A_{\alpha_i^\sigma, \alpha_j^\sigma} = A_{ij}$, so that $(A_{\alpha_i^\sigma, \alpha_j^\sigma})$ is nonsingular and indecomposable. Finally, β^σ is the unique *minimal* root relative to $\alpha_1^\sigma, \dots, \alpha_r^\sigma$, in the sense in which β is the unique minimal root relative to $\alpha_1, \dots, \alpha_r$. Therefore $\alpha_1^\sigma, \dots, \alpha_r^\sigma$ constitute a fundamental system of roots relative to $\mathfrak{H}\sigma$, and $(A_{\alpha_i^\sigma, \alpha_j^\sigma}) = (A_{ij})$.

If $e_\alpha \neq 0$ is a root-vector ($\alpha \neq 0$) relative to some standard Cartan subalgebra \mathfrak{H} of \mathfrak{L} , then $\text{ad}(e_\alpha): x \rightarrow [xe_\alpha]$ is a derivation D of \mathfrak{L} , and $D^4 = 0$ (see [15]). Since the characteristic of \mathfrak{F} is not 2 or 3, $A = \exp(\lambda D)$ is defined by the exponential power series for each $\lambda \in \mathfrak{F}$, and A is an automorphism of \mathfrak{L} . For $p = 5$ and \mathfrak{L} of type G_2 , a special treatment is required to show that A is an automorphism; in all cases other than G_2 , $D^3 = 0$ and no special argument is required. The group of automorphisms generated by all these $\exp(\lambda \text{ad}(e_\alpha))$, where λ runs through \mathfrak{F} and where e_α runs through a set of nonzero root-vectors belonging to all nonzero roots relative to all standard Cartan subalgebras \mathfrak{H} of \mathfrak{L} , will be called the group of *invariant automorphisms* of \mathfrak{L} , and will be denoted by \mathfrak{I} . In [15] we have shown that if \mathfrak{F} is algebraically closed, and if \mathfrak{H}_1 and \mathfrak{H}_2 are any two standard Cartan subalgebras of \mathfrak{L} , then there is an invariant automorphism A such that $\mathfrak{H}_1 A = \mathfrak{H}_2$. In particular, if \mathfrak{H} is a standard Cartan subalgebra and σ is an automorphism of \mathfrak{L} , then $\mathfrak{H}\sigma = \mathfrak{H}A$ for some invariant automorphism A , and $\mathfrak{H}\sigma A^{-1} = \mathfrak{H}$. Thus, in the algebraically closed case, the use of invariant automorphisms reduces the study of automorphisms to the investigation of those automorphisms which map a certain standard Cartan subalgebra onto itself.

Let us summarize the results of this section.

THEOREM 1. *Let \mathfrak{L} be a simple Lie algebra of classical type, \mathfrak{H} a standard Cartan subalgebra of \mathfrak{L} , and $\alpha_1, \dots, \alpha_r$ a fundamental system of roots relative to \mathfrak{H} . Let σ be an automorphism of \mathfrak{L} . Then $\mathfrak{H}\sigma$ is a standard Cartan subalgebra of \mathfrak{L} , and if α^σ is defined by $\alpha^\sigma(h\sigma) = \alpha(h)$ for each $\alpha \in \mathfrak{H}^*$, then α^σ is a root relative to $\mathfrak{H}\sigma$ if and only if α is a root relative to \mathfrak{H} . The mapping $\alpha \rightarrow \alpha^\sigma$ is a linear isomorphism of \mathfrak{H}^* onto $(\mathfrak{H}\sigma)^*$, and $\alpha_1^\sigma, \dots, \alpha_r^\sigma$ form a fundamental system of roots relative to $\mathfrak{H}\sigma$ with $A_{\alpha_i^\sigma, \alpha_j^\sigma} = A_{\alpha_i, \alpha_j}$ for all i, j . If the base field is algebraically closed, there is an invariant automorphism A of \mathfrak{L} such that $\mathfrak{H}\sigma = \mathfrak{H}A$.*

II. Fundamental systems and the Weyl group. Let \mathfrak{L} , \mathfrak{H} , and $\alpha_1, \dots, \alpha_r$ be as in Theorem 1. For each i , $1 \leq i \leq r$, let h_i be that unique element of $[\mathfrak{L}_{-\alpha_i}, \mathfrak{L}_{\alpha_i}]$ such that $\alpha_i(h_i) = 2$ ($= 2 \cdot 1$, where 1 is the unity of \mathfrak{F}). That such a choice is possible and unique follows from (iv) and (xi) of (C). We define

a linear transformation $S_{\alpha_i} = S_i$ of \mathfrak{H}^* by $\phi S_i = \phi - \phi(h_i)\alpha_i$, $\phi \in \mathfrak{H}^*$. Evidently $S_i^2 = I$, the identity transformation. If α is any root, then

$$(1) \quad \alpha S_i = \alpha - \alpha(h_i)\alpha_i = \alpha - \bar{A}_{\alpha, i}\alpha_i = \alpha - A_{\alpha, i}\alpha_i,$$

where \bar{n} is the residue class, in the prime field of \mathfrak{F} , of the rational integer n . This equation follows from (xiv) of (C). But $\alpha - A_{\alpha, i}\alpha_i$ is a root, by the definition of $A_{\alpha, i} = A_{\alpha, \alpha_i}$ in (xiv) of (C); therefore each S_i permutes the roots relative to \mathfrak{H} . Since the roots span \mathfrak{H}^* , each S_i is completely determined by its effect on the roots alone. Thus the group \mathfrak{W} generated by the S_i may be regarded as a subgroup of the group of all permutations of the set of roots relative to \mathfrak{H} , and so \mathfrak{W} is a finite group. As in Theorem 16.1 of [16], we could show directly that \mathfrak{W} is independent of the chosen fundamental system $\alpha_1, \dots, \alpha_r$. However, this result will be a corollary of the result in [16] after we have proved Lemma 5 below. Meanwhile we shall anticipate this independence by referring to \mathfrak{W} simply as the *Weyl group* of \mathfrak{L} relative to \mathfrak{H} .

Next we shall establish for Lie algebras of classical type a certain one-to-one correspondence between the elements of the Weyl group and fundamental systems of roots relative to \mathfrak{H} . Following an idea developed by Curtis (Proposition 5 of [4]) for algebras with nondegenerate Killing forms, we shall achieve this objective by setting up an "isomorphism" between the sets of roots of two simple Lie algebras of classical type, perhaps over base fields of different characteristics, provided these two Lie algebras have the same matrix (A_{ij}) for certain fundamental systems. There is also associated with this identification an isomorphism of the respective Weyl groups *as transformation groups acting on the roots*. The desired results for \mathfrak{L} are then immediate from the corresponding ones for simple algebras over algebraically closed fields of characteristic zero.

Again let \mathfrak{L} be a simple Lie algebra of classical type, \mathfrak{H} a standard Cartan subalgebra, $\alpha_1, \dots, \alpha_r$ a fundamental system of roots relative to \mathfrak{H} . Then there is a unique minimal root δ relative to $\alpha_1, \dots, \alpha_r$, in the sense of I , and if β is any root, there is a sequence $\beta, \beta - \alpha_{i_1}, \beta - \alpha_{i_1} - \alpha_{i_2}, \dots, \beta - \alpha_{i_1} - \dots - \alpha_{i_k} = \delta$ of roots, $1 \leq i_j \leq r$ (0 is here regarded as a root). The non-negative rational integer $k = N(\beta)$ is uniquely determined by β , and the integer $L(\beta) = N(\beta) - N(0)$ is called the *level* of β . (These and the following are the results of §11 of (C)). The function L has the properties:

- (a) $L(\beta) = 0$ if and only if $\beta = 0$.
- (b) $L(\beta) = 1$ if and only if $\beta = \alpha_i$ for some i ; $L(\beta) = -1$ if and only if $\beta = -\alpha_i$ for some i .
- (c) If $L(\beta) > 0$, then $\beta - \alpha_i$ is a root for some i ; if $L(\beta) < 0$, then $\beta + \alpha_i$ is a root for some i .
- (d) If α, β , and $\alpha + \beta$ are roots, then $L(\alpha + \beta) = L(\alpha) + L(\beta)$.
- (e) $L(-\beta) = -L(\beta)$ for all roots β .

We shall also need:

(f) The matrix (A_{ij}) is nonsingular (§§10 and 8 of (C)) and:

(g) If $\alpha, \beta, \gamma, \dots, \rho, \zeta$ are roots relative to \mathfrak{G} , and if $\alpha + \beta, \alpha + \beta + \gamma, \dots, \alpha + \beta + \dots + \rho$ are roots, then if $\zeta \neq 0$, $A_{\alpha+\beta+\dots+\rho, \zeta} = A_{\alpha, \zeta} + \dots + A_{\rho, \zeta}$ (Lemma 3 of (C), Lemmas D and L of [12]).

LEMMA 1. *Every root β with $L(\beta) > 0$ is a term of a chain $\alpha_{i_1}, \alpha_{i_1} + \alpha_{i_2}, \dots, \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_s} = \beta$ ($1 \leq i_j \leq r$) where all terms are roots. The number $t(\beta, i)$ of times which α_i occurs in the sum $\alpha_{i_1} + \dots + \alpha_{i_s}$ is uniquely determined by β . Moreover, β is not a term of any chain of roots of the form $-\alpha_{i_1}, -\alpha_{i_1} - \alpha_{i_2}, \dots, -\alpha_{i_1} - \dots - \alpha_{i_u} = \beta$, $1 \leq i_j \leq r$.*

Proof. The existence of such a chain is readily proved by induction on $L(\beta)$. If $L(\beta) = 1$, then $\beta = \alpha_i$ for some i , and we have the result. If $L(\beta) > 1$, then by (c) and (d), $\beta - \alpha_i = \gamma$ is a root of level $L(\beta) - 1 > 0$ for some α_i . We may assume the existence of a chain of roots $\alpha_{i_1}, \alpha_{i_1} + \alpha_{i_2}, \dots, \alpha_{i_1} + \dots + \alpha_{i_s} = \gamma$. Thus we have the chain of roots $\alpha_{i_1}, \alpha_{i_1} + \alpha_{i_2}, \dots, \alpha_{i_1} + \dots + \alpha_{i_s}, \alpha_{i_1} + \dots + \alpha_{i_s} + \alpha_i = \beta$.

If in a second such chain, we have $\beta = \alpha_{j_1} + \dots + \alpha_{j_t}$, where each α_i occurs $t'(\beta, i)$ times, then by (g) above, we have

$$(2) \quad A_{\beta, \alpha_j} = \sum_{i=1}^r t(\beta, i) A_{ij} = \sum_{i=1}^r t'(\beta, i) A_{ij}, \quad 1 \leq j \leq r.$$

By (f), it follows that $t'(\beta, i) = t(\beta, i)$, $1 \leq i \leq r$.

From (b) and (d), $L(\beta) = s$. If $\beta = -\alpha_{i_1} - \dots - \alpha_{i_u}$ is also a term of a "negative" chain, then from (b) and (d), $L(\beta) = -u$, a contradiction. Hence Lemma 1 is proved.

Since $-\beta$ is a root if and only if β is a root, it follows by (e) and Lemma 1 that if $L(\beta) < 0$, then there is a chain $-\alpha_{i_1}, -\alpha_{i_1} - \alpha_{i_2}, \dots, -\alpha_{i_1} - \dots - \alpha_{i_s} = \beta$ of roots, where the number of times $-\alpha_i$ occurs is unique and is $t(-\beta, i)$, and that β is not a member of any "positive" chain. In this case, we set $t(\beta, i) = -t(-\beta, i) \leq 0$, $1 \leq i \leq r$. Then for every root β , there is a uniquely determined expression $\beta = \sum_{i=1}^r t(\beta, i) \alpha_i$, where the $t(\beta, i)$ satisfy the above conditions. We call this representation the *canonical representation* of β in terms of the fundamental system $\alpha_1, \dots, \alpha_r$.

LEMMA 2. *The set of all r -tuples of natural numbers $(t(\beta, 1), \dots, t(\beta, r))$, formed for all roots β , depends only on the matrix (A_{ij}) , and is independent of the base field.*

Proof. Let \mathfrak{M} be a second simple Lie algebra of classical type relative to a standard Cartan subalgebra \mathfrak{R} over a base field \mathfrak{E} (of characteristic different from 2, 3) and let $\alpha'_1, \dots, \alpha'_r$ be a fundamental system of roots relative to \mathfrak{R} such that $A_{\alpha'_i, \alpha'_j} = A_{\alpha_i, \alpha_j}$ for all i, j . Let β be a root of \mathfrak{L} relative to \mathfrak{G} , $\beta = \sum_i t(\beta, i) \alpha_i$ its canonical representation. By symmetry it is enough to show that there is a root β' of \mathfrak{M} relative to \mathfrak{R} with the canonical representa-

tion $\beta' = \sum_i t(\beta, i)\alpha'_i$. Moreover, it is enough to treat only the case $L(\beta) > 0$ (by the remarks immediately preceding this lemma).

Again we proceed by induction on $L(\beta)$. If $L(\beta) = 1$, then $\beta = \alpha_i$ for some i , $t(\beta, j) = \delta_{ij}$, and $\alpha'_i = \sum_j t(\beta, j)\alpha'_j$ is the canonical representation of the root α'_i of \mathfrak{M} . Now let $L(\beta) > 1$, and assume the result for all roots of \mathfrak{L} of positive level less than $L(\beta)$. By (C), $\beta - \alpha_i$ is a root for some α_i , and $\beta \neq \alpha_i$ since $L(\beta) > 1$. Let m be the smallest positive integer such that $\beta - (m+1)\alpha_i$ is not a root of \mathfrak{L} . Then $1 \leq m \leq 3$, the second inequality by the fact that $\beta, \beta - \alpha_i, \beta - 2\alpha_i, \beta - 3\alpha_i, \beta - 4\alpha_i$ cannot all be roots ((v) and (xvii) of (C)). Now $L(\beta - m\alpha_i) = L(\beta) - m < L(\beta)$, and $L(\beta - m\alpha_i) > 0$ since if $L(\beta - m\alpha_i) \leq 0$, then either $L(\beta - 2\alpha_i) = 0$ or $L(\beta - 3\alpha_i) = 0$, and $\beta = 2\alpha_i$ or $\beta = 3\alpha_i$. By (xvi) of (C), neither of these can be the case, and $L(\beta - m\alpha_i) > 0$. If $\alpha = \beta - m\alpha_i$, we have $t(\alpha, j) = t(\beta, j) - m\delta_{ij}$, since if $\alpha_{i_1}, \dots, \alpha_{i_1} + \dots + \alpha_{i_n} = \alpha$ is a chain for α , the chain $\alpha_{i_1}, \dots, \alpha_{i_1} + \dots + \alpha_{i_n}, \alpha_{i_1} + \dots + \alpha_{i_n} + \alpha_i, \dots, \alpha_{i_1} + \dots + \alpha_{i_n} + m\alpha_i = \beta$ contains all α_j for $j \neq i$ precisely $t(\alpha, j)$ times and contains α_i exactly $t(\alpha, i) + m$ times. By assumption, there is a root α' of \mathfrak{M} with canonical representation $\alpha' = \sum_j t(\alpha, j)\alpha'_j$. But then by (g) we have

$$(3) \quad A_{\alpha, \alpha'_i} = \sum_j t(\alpha, j) A_{\alpha'_j, \alpha'_i} = \sum_j t(\alpha, j) A_{ji} = A_{\alpha, \alpha_i} \leq -m.$$

Thus $\alpha', \alpha' + \alpha'_i, \dots, \alpha' + m\alpha'_i = \beta'$ are roots of \mathfrak{M} , and $t(\beta', j) = t(\alpha', j) = t(\alpha, j) = t(\beta, j)$ if $j \neq i$, $t(\beta', i) = t(\alpha', i) + m = t(\alpha, i) + m = t(\beta, i)$. Thus β' has the canonical representation $\beta' = \sum_j t(\beta, j)\alpha'_j$, and Lemma 2 is proved by induction. (It will be noted that Lemma 2 is essentially proved in [14] by explicit computation of the roots.)

LEMMA 3. Let \mathfrak{L} and \mathfrak{M} be simple Lie algebras of classical type over fields $\mathfrak{F}, \mathfrak{E}$, respectively, with standard Cartan subalgebras $\mathfrak{H}, \mathfrak{R}$ respectively. Let $\alpha_1, \dots, \alpha_r$ be a fundamental system of roots for \mathfrak{L} relative to \mathfrak{H} , $\alpha'_1, \dots, \alpha'_r$ a fundamental system of roots for \mathfrak{M} relative to \mathfrak{R} , and suppose that $A_{\alpha_i, \alpha_j} = A_{\alpha'_i, \alpha'_j} = A_{ij}$, $1 \leq i, j \leq r$. Then there is a 1-1 mapping $\beta \rightarrow \beta'$ of the set of roots of \mathfrak{L} relative to \mathfrak{H} onto the set of roots of \mathfrak{M} relative to \mathfrak{R} such that:

- (1) $\alpha_i \rightarrow \alpha'_i$, $1 \leq i \leq r$;
- (2) if $\beta \rightarrow \beta'$, then $(-\beta) \rightarrow -\beta'$;
- (3) if α, β , and $\alpha + \beta$ are roots, then $(\alpha + \beta)' = \alpha' + \beta'$; if α and β are roots, and if $\alpha' + \beta'$ is a root, then $(\alpha + \beta)' = \alpha' + \beta'$;
- (4) $A_{\alpha, \beta} = A_{\alpha', \beta'}$ for all α, β ;
- (5) $(\beta S_i)' = \beta' S_{\alpha'_i}$ for all roots β and all i , $1 \leq i \leq r$.

Proof. Let β be a root of \mathfrak{L} relative to \mathfrak{H} , $\beta = \sum_i t(\beta, i)\alpha_i$ its canonical representation. We have seen in Lemma 2 that $\beta \rightarrow \beta' = \sum_i t(\beta, i)\alpha'_i$ defines a 1-1 mapping of the set of roots of \mathfrak{L} relative to \mathfrak{H} onto the set of roots of \mathfrak{M} relative to \mathfrak{R} . Moreover, $\beta' = \sum_i t(\beta, i)\alpha'_i$ is the canonical representation of β' relative to $\alpha'_1, \dots, \alpha'_r$, and $\alpha_i \rightarrow \alpha'_i$, $1 \leq i \leq r$. Since $-\beta = \sum_i t(-\beta, i)\alpha_i \rightarrow \sum_i t(-\beta, i)\alpha'_i = -\sum_i t(\beta, i)\alpha'_i = -\beta'$, (2) is satisfied. If α, β , and $\alpha + \beta = \gamma$

are roots of \mathfrak{L} , let $\gamma = \sum t(\gamma, i)\alpha_i$ be the canonical representation of γ . Then $\gamma' = \sum t(\gamma, i)\alpha'_i$ is the canonical representation of a root of \mathfrak{M} , as are $\alpha' = \sum t(\alpha, i)\alpha'_i$ and $\beta' = \sum t(\beta, i)\alpha'_i$. Since $\gamma = \alpha + \beta$, we see by (g) that

$$(4) \quad A_{\gamma, \alpha_j} = \sum_i t(\gamma, i) A_{ij} = A_{\alpha, \alpha_j} + A_{\beta, \alpha_j} = \sum_i (t(\alpha, i) + t(\beta, i)) A_{ij},$$

$1 \leq j \leq r$. Hence $t(\gamma, i) = t(\alpha, i) + t(\beta, i)$, $1 \leq i \leq r$, by (f), and $\alpha' + \beta' = \gamma' = (\alpha + \beta)'$. A completely analogous argument establishes the converse part of (3).

The result (4) follows from (3): if $\alpha - q\beta, \dots, \alpha - \beta, \alpha, \alpha + \beta, \dots, \alpha + m\beta$ are roots, and $\alpha - (q+1)\beta$ and $\alpha + (m+1)\beta$ are not roots, where m and q are non-negative integers, the corresponding assertion holds for the sequence $\alpha' - q\beta', \dots, \alpha' - \beta', \alpha', \alpha' + \beta', \dots, \alpha' + m\beta'$. Thus $A_{\alpha, \beta} = q - m = A_{\alpha', \beta'}$. Finally, $\beta S_i = \beta - A_{\beta, \alpha_i} \alpha_i$ is a root; it will be enough to prove (5) in case $L(\beta) > 0$. First let $\beta = \alpha_i$; then $\beta S_i = -\alpha_i$ and $(\beta S_i)' = -\alpha'_i = \alpha'_i S_{\alpha'_i} = \beta' S_{\alpha'_i}$. If $\beta \neq \alpha_i$, let $A_{\beta, \alpha_i} \geq 0$; then $L(\beta S_i) > 0$ since β is not a multiple of α_i . Now $\beta S_i, \beta S_i + \alpha_i, \dots, \beta S_i + A_{\beta, \alpha_i} \alpha_i = \beta$ are all roots, yielding $t(\beta, j) = t(\beta S_i, j) + A_{\beta, \alpha_i} \delta_{ij}$, $t(\beta S_i, j) = t(\beta, j) - A_{\beta, \alpha_i} \delta_{ij} = t(\beta', j) - A_{\beta', \alpha'_i} \delta_{ij} = t(\beta' S_{\alpha'_i}, j)$ by the same reasoning, $1 \leq i, j \leq r$. Therefore $(\beta S_i)' = \beta' S_{\alpha'_i}$, $1 \leq i \leq r$. An analogous argument proves $(\beta S_i)' = \beta' S_{\alpha'_i}$ if $A_{\beta, \alpha_i} < 0$. Thus Lemma 3 is proved.

LEMMA 4. *Let the notations be as in Lemma 3. Then if β_1, \dots, β_r is any fundamental system of roots for \mathfrak{L} relative to \mathfrak{S} , and if $\beta'_1, \dots, \beta'_r$ are their images under the mapping $\beta \rightarrow \beta'$, then $\beta'_1, \dots, \beta'_r$ is a fundamental system of roots for \mathfrak{M} relative to \mathfrak{R} , and $A_{\beta_i, \beta_j} = A_{\beta'_i, \beta'_j}$ for all i, j . Conversely, if $\beta'_1, \dots, \beta'_r$ is a fundamental system relative to \mathfrak{R} , then the preimages β_1, \dots, β_r constitute a fundamental system relative to \mathfrak{S} , and $A_{\beta_i, \beta_j} = A_{\beta'_i, \beta'_j}$ for all i, j .*

Proof. By the symmetry with its inverse of the mapping $\beta \rightarrow \beta'$, and by (4) of Lemma 3, it is enough to show that if β_1, \dots, β_r is a fundamental system, then so is $\beta'_1, \dots, \beta'_r$. Now if β_1, \dots, β_r is fundamental, then by Lemma 1 every nonzero root β relative to \mathfrak{S} occurs either in a chain of roots $\beta_{i_1}, \beta_{i_1} + \beta_{i_2}, \dots, \beta_{i_1} + \dots + \beta_{i_s} = \beta$ or in a chain $-\beta_{i_1}, -\beta_{i_1} - \beta_{i_2}, \dots, -\beta_{i_1} - \dots - \beta_{i_s} = \beta$, $1 \leq i_j \leq r$. Since the images β' exhaust all roots relative to \mathfrak{R} , it follows from (2) and (3) of Lemma 3 that every nonzero root β' relative to \mathfrak{R} occurs either in a chain of roots $\beta'_{i_1}, \beta'_{i_1} + \beta'_{i_2}, \dots, \beta'_{i_1} + \dots + \beta'_{i_s} = \beta'$ or in a chain $-\beta'_{i_1}, -\beta'_{i_1} - \beta'_{i_2}, \dots, -\beta'_{i_1} - \dots - \beta'_{i_s} = \beta'$. Thus $\beta'_1, \dots, \beta'_r$ spans \mathfrak{R}^* , since the roots span \mathfrak{R}^* . Next, β_1, \dots, β_r is indecomposable and (A_{β_i, β_j}) is nonsingular; since $A_{\beta_i, \beta_j} = A_{\beta'_i, \beta'_j}$, the same properties hold for $\beta'_1, \dots, \beta'_r$. If $\beta'_i - \beta'_j$ were a root for some $i \neq j$, then the preimage $\beta_i - \beta_j$ would be a root, contrary to assumption. Finally, let δ be that unique root relative to \mathfrak{S} such that no $\delta - \beta_i$ is a root. Then no $\delta' - \beta'_i$, $1 \leq i \leq r$, is a root relative to \mathfrak{R} , and δ' is a root relative to \mathfrak{R} with the same property of minimality. If γ' were a second root relative to \mathfrak{R} such that no $\gamma' - \beta'_i$ is a root, then no $\gamma - \beta_i$ would be a root, where γ is the preimage of

γ' . By the uniqueness of δ , $\gamma = \delta$, and $\gamma' = \delta'$. Thus the proof of Lemma 4 is complete.

LEMMA 5. *With the notations of Lemmas 2 and 3, the mappings $\beta \rightarrow \beta'$ and $S_{\alpha_i} \rightarrow S_{\alpha'_i}$ define an isomorphism of the Weyl group \mathfrak{W} relative to \mathfrak{S} (generated by the S_{α_i}), regarded as a transformation group on the roots relative to \mathfrak{S} , onto the Weyl group \mathfrak{W}' relative to \mathfrak{R} (generated by the $S_{\alpha'_i}$), regarded as a transformation group on the roots relative to \mathfrak{R} . That is, the mapping $S_{\alpha_i} \rightarrow S_{\alpha'_i}$ can be uniquely extended to an isomorphism τ of \mathfrak{W} onto \mathfrak{W}' such that for every root β relative to \mathfrak{S} and for every $S \in \mathfrak{W}$, $(\beta S)' = \beta' S'$.*

Proof. From (5) of Lemma 3, $(\beta S_{\alpha_i})' = \beta' S_{\alpha'_i}$ for all roots β . It follows that, in general, $(\beta S_{\alpha_{i_1}} \cdots S_{\alpha_{i_k}})' = \beta' S_{\alpha'_{i_1}} \cdots S_{\alpha'_{i_k}}$ for all β and for $1 \leq i_j \leq r$. Thus if $S_{\alpha_{i_1}} \cdots S_{\alpha_{i_k}} = S_{\alpha_{j_1}} \cdots S_{\alpha_{j_m}}$ then $\beta' S_{\alpha'_{i_1}} \cdots S_{\alpha'_{i_k}} = \beta' S_{\alpha'_{j_1}} \cdots S_{\alpha'_{j_m}}$ for all roots β' relative to \mathfrak{R} , so that the mapping $\tau: (S_{\alpha'_{i_1}} \cdots S_{\alpha'_{i_k}})' = S_{\alpha'_{i_1}} \cdots S_{\alpha'_{i_k}}$ defines a homomorphism of \mathfrak{W} onto \mathfrak{W}' . By symmetry, the mapping $\omega: (S_{\alpha'_{i_1}} \cdots S_{\alpha'_{i_k}})^\omega = S_{\alpha_{i_1}} \cdots S_{\alpha_{i_k}}$ defines a homomorphism of \mathfrak{W}' onto \mathfrak{W} such that $\tau\omega$ is the identity on \mathfrak{W} , $\omega\tau$ the identity on \mathfrak{W}' . Hence τ is an isomorphism with inverse ω , and τ is the only possible homomorphic extension to \mathfrak{W} of the mapping $S_{\alpha_i} \rightarrow S_{\alpha'_i}$. By the above, we have $(\beta S)' = \beta' S'$ for all roots β relative to \mathfrak{S} and for all $S \in \mathfrak{W}$. This completes the proof.

THEOREM 2. *Let \mathfrak{L} be a simple Lie algebra of classical type over a field \mathfrak{F} of characteristic not 2 or 3. Let \mathfrak{S} be a standard Cartan subalgebra of \mathfrak{L} and $\alpha_1, \dots, \alpha_r$ a fundamental system of roots relative to \mathfrak{S} . Let \mathfrak{W} be the group generated by the S_{α_i} . Then \mathfrak{W} is independent of the particular fundamental system $\alpha_1, \dots, \alpha_r$ chosen, and if β_1, \dots, β_s is a second fundamental system relative to \mathfrak{S} , then $s = r$ and there is a unique $S \in \mathfrak{W}$ such that β_1, \dots, β_r is a permutation of $\alpha_1 S, \dots, \alpha_r S$. For every $S \in \mathfrak{W}$, $\alpha_1 S, \dots, \alpha_r S$ is a fundamental system relative to \mathfrak{S} .*

Proof. First let \mathfrak{L} be a simple Lie algebra over an algebraically closed field of characteristic zero. It should first be noted that our notion of fundamental system of roots then coincides with the traditional notion of a fundamental system, namely, a linearly independent set of roots such that every root is a linear combination of this set with nonpositive integral coefficients or a linear combination of this set with non-negative integral coefficients. For by Lemma 24 of (C), a fundamental system in the sense of (C) is a basis for \mathfrak{S}^* if the base field is of characteristic zero, and in this case every root is either a non-negative or nonpositive integral combination of the system by Lemma 1 above. Conversely, if \mathfrak{F} is algebraically closed of characteristic zero and if $\alpha_1, \dots, \alpha_r$ is a fundamental system for \mathfrak{L} in the traditional sense, then the matrix $(\alpha_i(h_j)) = (A_{ij} \cdot 1)$ is known to be indecomposable and nonsingular (see [5; 6; 16]), and $\alpha_1, \dots, \alpha_r$ spans \mathfrak{S}^* by definition. Since \mathfrak{F} is of characteristic zero, and since there are only finitely many roots relative to \mathfrak{S} ,

no string of roots $\beta, \beta - \alpha_{i_1}, \dots, \beta - \alpha_{i_1} - \alpha_{i_2} - \dots - \alpha_{i_r}, \dots$ can continue indefinitely. Hence there must be a root δ such that no $\delta - \alpha_i$ is a root. But then δ is the highest weight of the adjoint representation of \mathfrak{L} on itself relative to the fundamental system $-\alpha_1, -\alpha_2, \dots, -\alpha_r$ (in the traditional sense). Since this representation is irreducible by the simplicity of \mathfrak{L} , no other weight (root, in this case) γ can have the property that no $\gamma - \alpha_i$ is a root (see [1; 16; 18]). Thus the two senses of fundamental system are equivalent for \mathfrak{L} .

Let Λ be a linear function on \mathfrak{G} such that $\Lambda(h_i) = m_i \cdot 1$, where the m_i , $1 \leq i \leq r$, are distinct positive integers. Then there is an irreducible representation of \mathfrak{L} with highest weight Λ (relative to $\alpha_1, \dots, \alpha_r$; see [1; 8; 16; 18] for the results on representations used in this paragraph). Since each $S \in \mathfrak{B}$ is linear, nonsingular, and permutes the roots, it follows that $\alpha_1 S, \dots, \alpha_r S$ is a fundamental system for each $S \in \mathfrak{B}$. Conversely, if β_1, \dots, β_s is a fundamental system, then $s = \dim \mathfrak{G} = r$, and our irreducible representation of \mathfrak{L} has a unique highest weight M relative to β_1, \dots, β_r . Then M is an extreme weight in the sense of §4 of [1], from which it follows that there is an $S \in \mathfrak{B}$ such that $MS = \Lambda$. Now $MS = \Lambda$ is the highest weight relative to $\beta_1 S, \dots, \beta_r S$. If $\gamma = \beta_i S$ is not a positive root (relative to $\alpha_1, \dots, \alpha_r$) for some i , then $\gamma = \sum n_j \alpha_j$, n_j integers ≤ 0 , some $n_j < 0$. Let $h \in [L_{-\gamma} L_{\gamma}]$, $\gamma(h) = 2$. Then $h = \sum \rho_j h_j$, $\rho_j \in \mathfrak{F}$. From $\alpha_k(h) = 2(\alpha_k, \gamma)(\gamma, \gamma)^{-1}$, where the scalar product is that induced on \mathfrak{G}^* by the Killing form on \mathfrak{G} , we obtain

$$\begin{aligned} \alpha_k(h) &= \sum_j \rho_j A_{kj} = 2(\gamma, \gamma)^{-1} \sum_j n_j (\alpha_k, \alpha_j) \\ &= (\gamma, \gamma)^{-1} \sum_j n_j (\alpha_j, \alpha_j) A_{kj}, \quad 1 \leq k \leq r. \end{aligned}$$

By the nonsingularity of the matrix (A_{kj}) , $\rho_j = (\gamma, \gamma)^{-1} (\alpha_j, \alpha_j) n_j$, $1 \leq j \leq r$, and for all roots α , (α, α) is known to be a positive rational number (identifying the rationals under the only possible isomorphism with the prime field of \mathfrak{F}) [16]. From $0 \leq 2(\Lambda, \gamma)(\gamma, \gamma)^{-1} = \Lambda(h) = \sum_j \rho_j m_j$, we have $\sum_j m_j n_j (\alpha_j, \alpha_j) \geq 0$. But all $m_j > 0$ and all $(\alpha_j, \alpha_j) > 0$, so that $\sum_j m_j n_j (\alpha_j, \alpha_j) < 0$, a contradiction. It follows that each $\beta_i S$ is a positive root with respect to $\alpha_1, \dots, \alpha_r$, and conversely each α_i is positive with respect to $\beta_1 S, \dots, \beta_r S$. An elementary argument shows that this is impossible unless $\beta_1 S, \dots, \beta_r S$ is a permutation of $\alpha_1, \dots, \alpha_r$. Thus every fundamental system is an image of $\alpha_1, \dots, \alpha_r$ under \mathfrak{B} . If $S \in \mathfrak{B}$ permutes $\alpha_1, \dots, \alpha_r$, then $\Lambda S = \Lambda$ is the highest weight relative to $\alpha_1 S, \dots, \alpha_r S$ and if $\alpha_i S = \alpha_{i\phi}$, ϕ a permutation of $(1, \dots, r)$, we have $\Lambda S(h_{i\phi}) = \Lambda(h_i)$, all i , or $\Lambda(h_{i\phi}) = \Lambda(h_i)$, all i . Since all $\Lambda(h_i)$ are distinct, $i\phi = i$ for all i , and S is the identity. The uniqueness assertion of the theorem now follows in this case.

(The referee has pointed out that in case \mathfrak{F} is the complex field, a proof of the assertions of the theorem which is independent of some of the results essential to the argument above has been given by Satake [20]. For the next part of the proof one can take the field \mathfrak{C} to be the complex field and can ap-

ply Satake's result instead of the above.)

Now let \mathfrak{L} , \mathfrak{F} , \mathfrak{G} , $\alpha_1, \dots, \alpha_r$, \mathfrak{B} , and β_1, \dots, β_s be as in the statement of the theorem. Then there is a simple Lie algebra \mathfrak{M} over an algebraically closed field \mathfrak{C} of characteristic zero with standard Cartan subalgebra \mathfrak{K} and fundamental system of roots $\alpha'_1, \dots, \alpha'_r$ relative to \mathfrak{K} such that $A_{\alpha'_i, \alpha'_i} = A_{\alpha_i, \alpha_j}$, $1 \leq i, j \leq r$ (see Theorem 6 of (C) and, for example, [5; 8; 19]). Let \mathfrak{B}' be the Weyl group of \mathfrak{M} relative to \mathfrak{K} (generated by the $S_{\alpha'_i}$). Let the correspondence $\beta \rightarrow \beta'$ and the isomorphism τ of \mathfrak{B} onto \mathfrak{B}' be as in Lemmas 3-5.

If β is any root relative to \mathfrak{G} ($\beta \neq 0$) and γ is any other root, then $\gamma S_\beta = \gamma - A_{\gamma, \beta} \beta$. We wish to show $S_\beta \in \mathfrak{B}$. But $\gamma' S_{\beta'} = \gamma' - A_{\gamma', \beta'} \beta' = \gamma' - A_{\gamma, \beta} \beta'$ and $S_{\beta'} \in \mathfrak{B}'$ by our earlier remarks based on [16]. If $T = S_\beta^{-1}$, then $(\gamma T)' = \gamma' T' = \gamma' S_{\beta'} = \gamma' - A_{\gamma, \beta} \beta'$, so that $\gamma T = \gamma - A_{\gamma, \beta} \beta = \gamma S_\beta$ for all roots γ . Thus $S_\beta = T \in \mathfrak{B}$, and $S_\beta^r = S_{\beta'}$. It follows that \mathfrak{B} is independent of the fundamental system $\alpha_1, \dots, \alpha_r$ used in its generation. If β_1, \dots, β_s is a second fundamental system relative to \mathfrak{G} , then by Lemma 4, $\beta'_1, \dots, \beta'_s$ is a fundamental system relative to \mathfrak{K} . Hence $r=s$ by the results at characteristic zero, and there is a unique $U \in \mathfrak{B}'$ such that $\beta'_1, \dots, \beta'_r$ is a permutation of $\alpha'_1 U, \dots, \alpha'_r U$. Let $S = U^{r-1} \in \mathfrak{B}$. Then $(\alpha_i S)' = \alpha'_i S' = \alpha'_i U$, $1 \leq i \leq r$, and $(\alpha_1 S)', \dots, (\alpha_r S)'$ is a permutation of $\beta'_1, \dots, \beta'_r$. Hence $\alpha_1 S, \dots, \alpha_r S$ is a permutation of β_1, \dots, β_r . If $\alpha_1 T, \dots, \alpha_r T$ is also a permutation of β_1, \dots, β_r , where $T \in \mathfrak{B}$, then $(\alpha_1 T)', \dots, (\alpha_r T)'$ is a permutation of $\beta'_1, \dots, \beta'_r$, i.e., $\alpha'_1 T', \dots, \alpha'_r T'$ is a permutation of $\beta'_1, \dots, \beta'_r$. Hence $T' = U = S'$, and $T = S$. Finally if $S \in \mathfrak{B}$, then $S' \in \mathfrak{B}'$, and $\alpha'_1 S', \dots, \alpha'_r S'$ is a fundamental system relative to \mathfrak{K} . That is, $(\alpha_1 S)', \dots, (\alpha_r S)'$ is a fundamental system relative to \mathfrak{K} . By Lemma 4, $\alpha_1 S, \dots, \alpha_r S$ is a fundamental system relative to \mathfrak{G} , and the proof of Theorem 2 is complete.

III. Automorphisms mapping a Cartan subalgebra into itself. Let \mathfrak{L} be simple, \mathfrak{G} a standard Cartan subalgebra, $\alpha_1, \dots, \alpha_r$ a fundamental system of roots relative to \mathfrak{G} . Let h_i be as before, i.e., $h_i \in [\mathfrak{L}_{\alpha_i} \mathfrak{L}_{-\alpha_i}]$, $\alpha_i(h_i) = 2$, $1 \leq i \leq r$, and let $e_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$, $e_{-\alpha_i} \in \mathfrak{L}_{-\alpha_i}$, with $[e_{-\alpha_i}, e_{\alpha_i}] = h_i$. Let $0 \neq e_\beta \in \mathfrak{L}_\beta$ for each of the remaining roots β . Then the h_i span \mathfrak{G} , and the h_i together with all the e_α span \mathfrak{L} ((C), (xii) and Lemma 23). Also if $\alpha \neq -\gamma$, we have $[e_\alpha, e_\gamma] = N_{\alpha, \gamma} e_{\alpha+\gamma}$, where $N_{\alpha, \gamma} \in \mathfrak{F}$, $N_{\alpha, \gamma} \neq 0$ if and only if $\alpha + \gamma$ is a root ((C), (xii) and (xiii)). Let $\lambda \neq 0$ be an element of \mathfrak{F} , $D_i = \text{ad}(e_{\alpha_i})$. We compute the effect of the automorphism $A_i(\lambda) = \exp(\lambda D_i)$ on \mathfrak{G} .

Now \mathfrak{G} has a basis consisting of h_i and a set of vectors h such that $\alpha_i(h) = 0$. For the latter, we have $h D_i = [h e_{\alpha_i}] = -\alpha_i(h) e_{\alpha_i} = 0$, so that $h A_i(\lambda) = h$ if $\alpha_i(h) = 0$. Also, $h_i D_i = [h_i e_{\alpha_i}] = -\alpha_i(h_i) e_{\alpha_i} = -2e_{\alpha_i}$, and $h_i D_i^2 = -2[e_{\alpha_i}, e_{\alpha_i}] = 0$. Hence $h_i A_i(\lambda) = h_i - 2\lambda e_{\alpha_i}$. Further, we note that $e_{\alpha_i} A_i(\lambda) = e_{\alpha_i}$, and that

$$\begin{aligned} (5) \quad e_{-\alpha_i} A_i(\lambda) &= e_{-\alpha_i} + \lambda [e_{-\alpha_i}, e_{\alpha_i}] + \frac{\lambda^2}{2!} [[e_{-\alpha_i}, e_{\alpha_i}], e_{\alpha_i}] \\ &= e_{-\alpha_i} + \lambda h_i - \lambda^2 e_{\alpha_i}. \end{aligned}$$

Next let $B_i(\mu) = \exp(\mu \operatorname{ad}(e_{-\alpha_i}))$, $\mu \in \mathfrak{F}$. Then if $\alpha_i(h) = 0$, $hA_i(\lambda)B_i(\mu) = h$ as before. Also

$$\begin{aligned} h_i A_i(\lambda) B_i(\mu) &= (h_i - 2\lambda e_{\alpha_i}) B_i(\mu) \\ &= h_i B_i(\mu) - 2\lambda e_{\alpha_i} B_i(\mu) \\ &= h_i + 2\mu e_{-\alpha_i} - 2\lambda e_{\alpha_i} + 2\lambda\mu h_i + 2\lambda\mu^2 e_{-\alpha_i} \\ &= 2\mu(1 + \lambda\mu)e_{-\alpha_i} + (1 + 2\lambda\mu)h_i - 2\lambda e_{\alpha_i}. \end{aligned}$$

In particular, $h_i A_i(\lambda) B_i(-\lambda^{-1}) = -h_i - 2\lambda e_{\alpha_i}$, and

$$(6) \quad hA_i(\lambda)B_i(-\lambda^{-1})A_i(\lambda) = h \text{ if } \alpha_i(h) = 0;$$

$$(7) \quad h_i A_i(\lambda) B_i(-\lambda^{-1}) A_i(\lambda) = -h_i + 2\lambda e_{\alpha_i} - 2\lambda e_{\alpha_i} = -h_i.$$

Thus $T(\lambda, i) = A_i(\lambda)B_i(-\lambda^{-1})A_i(\lambda)$ is an invariant automorphism τ mapping \mathfrak{G} onto itself, with $h\tau = h$ if $\alpha_i(h) = 0$, and with $h_i\tau = -h_i$. If α is any root relative to \mathfrak{G} , α^τ is defined by $\alpha^\tau(h\tau) = \alpha(h)$ for all $h \in \mathfrak{G}$. Since $\mathfrak{G}\tau = \mathfrak{G}$, α^τ is also a root relative to \mathfrak{G} , with $\alpha^\tau(h) = \alpha(h)$ if $\alpha_i(h) = 0$ and with $\alpha^\tau(h_i) = -\alpha(h_i)$. If $\alpha_i(h) = 0$, we have $(\alpha S_i)(h) = (\alpha - \alpha(h_i)\alpha_i)(h) = \alpha(h)$, and we also have $(\alpha S_i)(h_i) = \alpha(h_i) - \alpha(h_i)\alpha_i(h_i) = \alpha(h_i) - 2\alpha(h_i) = -\alpha(h_i)$. Thus $\alpha^\tau = \alpha S_i$ for all roots α .

Now let σ be any automorphism of \mathfrak{L} mapping \mathfrak{G} onto \mathfrak{G} . Then $\alpha_1^\sigma, \dots, \alpha_r^\sigma$ is a fundamental system of roots relative to $\mathfrak{G}\sigma = \mathfrak{G}$ by Theorem 1. Therefore, by Theorem 2, there is a unique $S \in \mathfrak{B}$ such that $\alpha_1 S, \dots, \alpha_r S$ is a permutation of $\alpha_1^\sigma, \dots, \alpha_r^\sigma$. If $S = S_{i_1} \cdots S_{i_k}$, where $S_{i_j} = S_{\alpha_{i_j}}$, $1 \leq i_j \leq r$, and if λ_{i_j} , $1 \leq i_j \leq r$, are nonzero elements of \mathfrak{F} , let $\tau = T(\lambda_{i_1}, i_1) \cdots T(\lambda_{i_k}, i_k)$. Then τ is an invariant automorphism of \mathfrak{L} mapping \mathfrak{G} onto \mathfrak{G} , and $\alpha^\tau = \alpha S$ for all roots α . Thus $\alpha_1^\tau, \dots, \alpha_r^\tau$ are a permutation of $\alpha_1^\sigma, \dots, \alpha_r^\sigma$, and $\alpha_1^{\sigma\tau^{-1}}, \dots, \alpha_r^{\sigma\tau^{-1}}$ is a fundamental system which is a permutation of $\alpha_1, \dots, \alpha_r$. In addition, it follows from Theorem 1 that if $\beta_i = \alpha_i^{\sigma\tau^{-1}}$, $1 \leq i \leq r$, then $A_{\beta_i, \beta_j} = A_{\alpha_i, \alpha_j}$, $1 \leq i, j \leq r$. Thus we have proved the following theorem:

THEOREM 3. *Let \mathfrak{L} be a simple Lie algebra of classical type, \mathfrak{G} a standard Cartan subalgebra, $\alpha_1, \dots, \alpha_r$ a fundamental system of roots relative to \mathfrak{G} . Let σ be an automorphism of \mathfrak{L} such that $\mathfrak{G}\sigma = \mathfrak{G}$. Then there is an invariant automorphism τ of \mathfrak{L} such that $\mathfrak{G}\tau = \mathfrak{G}$, and such that if $\beta_i = \alpha_i^{\sigma\tau^{-1}}$, $1 \leq i \leq r$, then β_1, \dots, β_r is a permutation of $\alpha_1, \dots, \alpha_r$, with $A_{\beta_i, \beta_j} = A_{\alpha_i, \alpha_j}$, $1 \leq i, j \leq r$.*

Next let $\alpha_1, \dots, \alpha_r$ and β_1, \dots, β_r be as in Theorem 3, with $\beta_i = \alpha_{i\phi}$, $\phi \in \Sigma_r$, the symmetric group on r letters. Then $A_{i\phi, j\phi} = A_{ij}$, $1 \leq i, j \leq r$. That is, the matrix with $A_{i\phi, j\phi}$ as the entry in the i th row and j th column is the matrix (A_{ij}) , or the Schläfli diagram of the (ordered) system $\alpha_{1\phi}, \dots, \alpha_{r\phi}$ is the same as that of $\alpha_1, \dots, \alpha_r$. Thus $\alpha_i \rightarrow \alpha_{i\phi}$ effects a symmetry of this diagram (or of the corresponding matrix). (The diagrams by themselves do not display completely the matrices (A_{ij}) ; in case a triple or double line is included, one must also consider the integers corresponding to its pair of endpoints.) Examination of the matrices of Table 2 of [6] (see note below), or of the

diagrams of (C), shows immediately that $i = i\phi$ for all i except in case $\alpha_1, \dots, \alpha_r$ is of one of the types A_r ($r \geq 2$), D_r ($r \geq 4$), E_6 . If $\alpha_1, \dots, \alpha_r$ is of type A_r , and is labeled as in [6] or in (C), we have either $i\phi = i$ for all i , or $i\phi = r+1-i$ for all i . Similarly, if $\alpha_1, \dots, \alpha_r$ is of type D_r ($r \geq 5$), then either $i\phi = i$ for all i , or $i\phi = i$ for all $i \leq r-2$, and $(r-1)\phi = r$, $r\phi = r-1$. If $\alpha_1, \dots, \alpha_6$ is of type E_6 , then either $i\phi = i$ for all i , or $6\phi = 6$ and $i\phi = 6-i$, $1 \leq i \leq 5$. If $\alpha_1, \dots, \alpha_4$ is of type D_4 , then $2\phi = 2$, and $1\phi, 3\phi, 4\phi$ may be any one of the six possible permutations of 1, 3, 4.

NOTE. In Table 2 of [6], we have $A_{ij} = g^{ij}$ for all i, j in all cases except the following: In B_n , $A_{nj} = g^{nj}/2$, $1 \leq j \leq n$; in C_n , $A_{ij} = g^{ij}/2$, $1 \leq i \leq n-1$, $1 \leq j \leq n$; in G_2 , $A_{2j} = g^{2j}/3$, $j = 1, 2$; in F_4 , $A_{ij} = g^{ij}/2$, $1 \leq j \leq 4$, $i = 3, 4$. The inverse matrix (B_{ij}) to (A_{ij}) coincides with Dynkin's matrix $(g_{\alpha\beta})$ except as follows: In B_n , $B_{in} = 2g_{in}$, $1 \leq i \leq n$; in C_n , $B_{ij} = 2g_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n-1$; in G_2 , $B_{i2} = 3g_{i2}$, $i = 1, 2$; in F_4 , $B_{ij} = 2g_{ij}$, $1 \leq i \leq 4$, $j = 3, 4$.

In any case, we have $e_{\alpha_i}\sigma\tau^{-1} = \mu_i e_{\beta_i}$ and $e_{-\alpha_i}\sigma\tau^{-1} = \nu_i e_{-\beta_i}$, $1 \leq i \leq r$, where $\mu_i \neq 0$, $\nu_i \neq 0$. $[e_{-\alpha_i}, e_{\alpha_i}]\sigma\tau^{-1} = h_i\sigma\tau^{-1}$ and

$$\alpha_i\phi(h_i\sigma\tau^{-1}) = \alpha_i\sigma\tau^{-1}(h_i\sigma\tau^{-1}) = \alpha_i(h_i) = 2.$$

Moreover

$$[e_{-\alpha_i}, e_{\alpha_i}]\sigma\tau^{-1} = [e_{-\alpha_i}\sigma\tau^{-1}, e_{\alpha_i}\sigma\tau^{-1}] = \nu_i\mu_i[e_{-\beta_i}, e_{\beta_i}] \in [\mathfrak{L}_{-\beta_i}, \mathfrak{L}_{\beta_i}].$$

Hence

$$[e_{-\alpha_i}, e_{\alpha_i}]\sigma\tau^{-1} = h_{i\phi} = [e_{-\beta_i}, e_{\beta_i}] = \nu_i\mu_i[e_{-\beta_i}, e_{\beta_i}],$$

$\nu_i\mu_i = 1$, $\nu_i = \mu_i^{-1}$. Thus we see that $\omega = \sigma\tau^{-1}$ is completely determined on the $e_{-\alpha_i}$ by the nonzero scalars μ_1, \dots, μ_r . Since the e_{α_i} and the $e_{-\alpha_i}$ generate \mathfrak{L} (by Lemma 1 above, (xii), (xiii) and Lemma 23 of (C)), ω is completely determined on \mathfrak{L} by the nonzero scalars μ_1, \dots, μ_r . Summarizing, we have:

LEMMA 6. *Let ω be an automorphism mapping \mathfrak{L} onto \mathfrak{L} and such that $\alpha_i^\omega = \alpha_{i\phi}$, $1 \leq i \leq r$, where $\phi \in \Sigma_r$. Then ω is completely determined by the permutation ϕ and the scalars μ_1, \dots, μ_r .*

IV. **Automorphisms leaving a Cartan subalgebra fixed.** We begin by investigating more closely the invariant automorphisms $\tau = T(\lambda, i)$ for $\lambda \neq 0$. We have

$$\begin{aligned} e_{\alpha_i}\tau &= e_{\alpha_i}B_i(-\lambda^{-1})A_i(\lambda) = (e_{\alpha_i} + \lambda^{-1}h_i - \lambda^{-2}e_{-\alpha_i})A_i(\lambda) \\ (8) \quad &= e_{\alpha_i} + \lambda^{-1}h_i - 2e_{\alpha_i} - \lambda^{-2}e_{-\alpha_i} - \lambda^{-1}h_i + e_{\alpha_i} = -\lambda^{-2}e_{-\alpha_i}, \end{aligned}$$

$$\begin{aligned} e_{-\alpha_i}\tau &= (e_{-\alpha_i} + \lambda h_i - \lambda^2 e_{\alpha_i})B_i(-\lambda^{-1})A_i(\lambda) \\ (9) \quad &= (e_{-\alpha_i} + \lambda h_i - 2e_{-\alpha_i} - \lambda^2 e_{\alpha_i} - \lambda h_i + e_{-\alpha_i})A_i(\lambda) \\ &= -\lambda^2 e_{\alpha_i}A_i(\lambda) = -\lambda^2 e_{\alpha_i}. \end{aligned}$$

Let β be a root, $\beta \neq \pm \alpha_i$, $\beta - \alpha_i$ not a root, $\beta + \alpha_i$ a root.

CASE 1. $A_{\beta, \alpha_i} = -1$. Here $e_{\beta\tau}$ belongs to the root $\beta\tau = \beta S_i = \beta + \alpha_i$; $\beta + 2\alpha_i$ is not a root, and we have

$$\begin{aligned} e_{\beta\tau} &= e_{\beta} A_i(\lambda) B_i(-\lambda^{-1}) A_i(\lambda) = (e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i}) B_i(-\lambda^{-1}) A_i(\lambda) \\ &= (e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i} - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_{\beta}) A_i(\lambda) \\ &= (1 - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i}) e_{\beta} \\ &\quad + \lambda(1 - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i}) N_{\beta, \alpha_i} e_{\beta + \alpha_i} + \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i}. \end{aligned}$$

Since $e_{\beta\tau} \in \mathfrak{g}_{\beta + \alpha_i}$, the coefficient $1 - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i}$ of e_{β} must be zero; that is,

$$(10) \quad N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i} = 1,$$

and

$$(11) \quad e_{\beta\tau} = \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i}.$$

CASE 2. $A_{\beta, \alpha_i} = -2$. Here $e_{\beta\tau}$ belongs to $\beta S_i = \beta + 2\alpha_i$, $\beta + 3\alpha_i$ is not a root, and

$$\begin{aligned} e_{\beta\tau} &= \left(e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i} + \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} e_{\beta + 2\alpha_i} \right) B_i(-\lambda^{-1}) A_i(\lambda) \\ &= \left(e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i} - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_{\beta} + \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} e_{\beta + 2\alpha_i} \right. \\ &\quad \left. - \frac{\lambda}{2!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} e_{\beta + \alpha_i} \right. \\ &\quad \left. + \frac{1}{2!^2} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_{\beta} \right) A_i(\lambda). \end{aligned}$$

The coefficient of e_{β} in $e_{\beta\tau}$, when the latter is written as a linear combination of root vectors, will be the same as that in $e_{\beta} A_i(\lambda) B_i(-\lambda^{-1})$, namely

$$(12) \quad 1 - N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i} + \frac{1}{2!^2} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} N_{\beta + \alpha_i, -\alpha_i},$$

and therefore this will be zero. Therefore

$$\begin{aligned} e_{\beta\tau} &= \left(\lambda N_{\beta, \alpha_i} \left(1 - \frac{1}{2!} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} \right) e_{\beta + \alpha_i} \right. \\ &\quad \left. + \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} e_{\beta + 2\alpha_i} \right) A_i(\lambda). \end{aligned}$$

But now, as above, the coefficient of $e_{\beta + \alpha_i}$ in this expression must be zero, i.e.

$$(13) \quad N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, -\alpha_i} = 2.$$

From the fact that (12) is zero, we now conclude, using (13),

$$(14) \quad N_{\beta, \alpha_i} N_{\beta+\alpha_i, -\alpha_i} = 2,$$

and $e_{\beta\tau}$ is $(\lambda^2/2!) N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} e_{\beta+2\alpha_i} A_i(\lambda)$; that is,

$$(15) \quad e_{\beta\tau} = \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} e_{\beta+2\alpha_i}.$$

CASE 3. $A_{\beta, \alpha_i} = -3$. Here $e_{\beta\tau}$ belongs to $\beta+3\alpha_i$, $\beta+4\alpha_i$ is not a root, and

$$\begin{aligned} e_{\beta\tau} &= \left(e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta+\alpha_i} + \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} e_{\beta+2\alpha_i} \right. \\ &\quad \left. + \frac{\lambda^3}{3!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} e_{\beta+3\alpha_i} \right) B_i(-\lambda)^{-1} A_i(\lambda) \\ &= \left(e_{\beta} + \lambda N_{\beta, \alpha_i} e_{\beta+\alpha_i} - N_{\beta, \alpha_i} N_{\beta+\alpha_i, -\alpha_i} e_{\beta} \right. \\ &\quad + \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} e_{\beta+2\alpha_i} - \frac{\lambda}{2!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha} N_{\beta+2\alpha_i, -\alpha_i} e_{\beta+\alpha} \\ &\quad + \frac{1}{2!^2} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, -\alpha_i} N_{\beta+\alpha_i, -\alpha_i} e_{\beta} \\ &\quad + \frac{\lambda^3}{3!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} e_{\beta+3\alpha_i} \\ &\quad - \frac{\lambda^2}{3!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} N_{\beta+3\alpha_i, -\alpha_i} e_{\beta+2\alpha_i} \\ &\quad + \frac{\lambda}{2!3!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} N_{\beta+3\alpha_i, -\alpha_i} N_{\beta+2\alpha_i, -\alpha_i} e_{\beta+\alpha} \\ &\quad \left. - \frac{1}{3!^2} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} N_{\beta+3\alpha_i, -\alpha_i} N_{\beta+2\alpha_i, -\alpha_i} N_{\beta+\alpha_i, -\alpha_i} e_{\beta} \right) A_i(\lambda). \end{aligned}$$

As in Cases 1 and 2, we find from observing in turn that the coefficients of e_{β} , $e_{\beta+\alpha_i}$, $e_{\beta+2\alpha_i}$ in $e_{\beta\tau}$, hence in $e_{\beta} A_i(\lambda) B_i(-\lambda)^{-1}$, must be zero, that

$$(16) \quad N_{\beta+2\alpha_i, \alpha_i} N_{\beta+3\alpha_i, -\alpha_i} = 3,$$

$$(17) \quad N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, -\alpha_i} = 4,$$

$$(18) \quad N_{\beta, \alpha_i} N_{\beta+\alpha_i, -\alpha_i} = 3,$$

and finally that

$$(19) \quad e_{\beta\tau} = \frac{\lambda^3}{3!} N_{\beta, \alpha_i} N_{\beta+\alpha_i, \alpha_i} N_{\beta+2\alpha_i, \alpha_i} e_{\beta+3\alpha_i}.$$

Now let $\gamma \neq \pm \alpha_i$ be a root such that $\gamma + \alpha_i$ is not a root. If $\gamma - \alpha_i$ is not a root, then $e_\gamma \tau = e_\gamma T(\lambda, i) = e_\gamma$. Otherwise, we have three cases:

CASE 1'. $A_{\gamma, \alpha_i} = 1$. Here $\gamma^\tau = \gamma - \alpha_i$, $e_\gamma \tau \in \mathfrak{L}_{\gamma - \alpha_i}$, $\gamma - 2\alpha_i$ is not a root, and

$$\begin{aligned} e_\gamma \tau &= e_\gamma A_i(\lambda) B_i(-\lambda^{-1}) A_i(\lambda) = e_\gamma B_i(-\lambda^{-1}) A_i(\lambda) \\ &= (e_\gamma - \lambda^{-1} N_{\gamma, -\alpha_i} e_{\gamma - \alpha_i}) A_i(\lambda) \\ &= -\lambda^{-1} N_{\gamma, -\alpha_i} e_{\gamma - \alpha_i} + (1 - N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, \alpha_i}) e_\gamma. \end{aligned}$$

As in Case 1, we have

$$(20) \quad N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, \alpha_i} = 1,$$

and

$$(21) \quad e_\gamma \tau = -\lambda^{-1} N_{\gamma, -\alpha_i} e_{\gamma - \alpha_i}.$$

CASE 2'. $A_{\gamma, \alpha_i} = 2$. The procedures established in the preceding cases yield

$$(22) \quad N_{\gamma - \alpha_i, -\alpha_i} N_{\gamma - 2\alpha_i, \alpha_i} = 2,$$

$$(23) \quad N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, \alpha_i} = 2,$$

and

$$(24) \quad e_\gamma \tau = \frac{\lambda^{-2}}{2!} N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, -\alpha_i} e_{\gamma - 2\alpha_i}.$$

CASE 3'. $A_{\gamma, \alpha_i} = 3$. As in previous cases, we find

$$(25) \quad N_{\gamma - 2\alpha_i, -\alpha_i} N_{\gamma - 3\alpha_i, \alpha_i} = 3,$$

$$(26) \quad N_{\gamma - \alpha_i, -\alpha_i} N_{\gamma - 2\alpha_i, \alpha_i} = 4,$$

$$(27) \quad N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, \alpha_i} = 3,$$

and

$$(28) \quad e_\gamma \tau = \frac{-\lambda^{-3}}{3!} N_{\gamma, -\alpha_i} N_{\gamma - \alpha_i, -\alpha_i} N_{\gamma - 2\alpha_i, -\alpha_i} e_{\gamma - 3\alpha_i}.$$

In particular, the primed cases apply when β is as in Cases 1, 2, 3 and $\gamma = \beta S_i = \beta - A_{\beta, \alpha_i} \alpha_i$. Then $A_{\gamma, \alpha_i} = A_{\beta, \alpha_i} - A_{\beta, \alpha_i} A_{\alpha_i, \alpha_i} = A_{\beta, \alpha_i} - 2A_{\beta, \alpha_i} = -A_{\beta, \alpha_i}$. Thus if β is as in Case 1, 2, 3, then $\gamma = \beta S_i$ is as in Case 1', 2', 3', respectively, and the Equations (20), (22), (23), (25), (26), (27) are corollaries of the respective Equations (10), (14), (13), (18), (17), (16).

Now let $\omega = U(\lambda, i) = T(\lambda, i)T(-1, i)$. Then if $\alpha_i(h) = 0$, we have $h\omega = h$, and $h_i\omega = -h_iT(-1, i) = h_i$. Thus ω is the identity on \mathfrak{H} . Moreover, $\alpha^\omega = (\alpha S_i)S_i = \alpha S_i^2 = \alpha$ for all roots α . By Lemma 6, ω is completely determined by its effect on the e_{α_j} , $1 \leq j \leq r$. By (8) and (9), we have

$$\begin{aligned} (29) \quad e_{\alpha_i} \omega &= e_{\alpha_i} T(\lambda, i) T(-1, i) = -\lambda^{-2} e_{-\alpha_i} T(-1, i) = \lambda^{-2} e_{\alpha_i}; \\ e_{\alpha_i} \omega &= \lambda^{-2} e_{\alpha_i} = \lambda^{-A_{ii}} e_{\alpha_i}. \end{aligned}$$

If $\beta \neq \pm \alpha_i$, and if $\beta - \alpha_i$ is not a root, we have:

For $A_{\beta, \alpha_i} = 0$:

$$(30) \quad e_\beta \omega = e_\beta = \lambda^{-A_{\beta, \alpha_i}} e_\beta$$

by the remark following (19).

For $A_{\beta, \alpha_i} = -1$:

$$e_\beta \omega = \lambda N_{\beta, \alpha_i} e_{\beta + \alpha_i} T(-1, i) = \lambda N_{\beta, \alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_\beta = \lambda e_\beta,$$

by (11), (21), and (10). Thus

$$(31) \quad e_\beta \omega = \lambda e_\beta = \lambda^{-A_{\beta, \alpha_i}} e_\beta.$$

For $A_{\beta, \alpha_i} = -2$:

$$\begin{aligned} e_\beta \omega &= \frac{\lambda^2}{2!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} e_{\beta + 2\alpha_i} T(-1, i) \\ &= \frac{\lambda^2}{2!^2} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_\beta, \end{aligned}$$

by (15) and (24). Now, applying (13) and (14), we have

$$(32) \quad e_\beta \omega = \lambda^2 e_\beta = \lambda^{-A_{\beta, \alpha_i}} e_\beta.$$

For $A_{\beta, \alpha_i} = -3$:

$$\begin{aligned} e_\beta \omega &= \frac{\lambda^3}{3!} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, \alpha_i} e_{\beta + 3\alpha_i} T(-1, i) \\ &= \frac{\lambda^3}{3!^2} N_{\beta, \alpha_i} N_{\beta + \alpha_i, \alpha_i} N_{\beta + 2\alpha_i, \alpha_i} N_{\beta + 3\alpha_i, -\alpha_i} N_{\beta + 2\alpha_i, -\alpha_i} N_{\beta + \alpha_i, -\alpha_i} e_\beta, \end{aligned}$$

by (19) and (28). Using (16), (17), and (18), we obtain

$$(33) \quad e_\beta \omega = \lambda^3 e_\beta = \lambda^{-A_{\beta, \alpha_i}} e_\beta.$$

From (29)–(33), we see that for all j , $1 \leq j \leq r$,

$$(34) \quad e_{\alpha_j} \omega = \lambda^{-A_{j i}} e_{\alpha_j}.$$

Now if $\lambda_1, \dots, \lambda_r$ are nonzero elements of \mathfrak{F} , and if $\eta = U(\lambda_1, 1) \dots U(\lambda_r, r)$, we have $h\eta = \eta$ for all $h \in \mathfrak{H}$, and

$$(35) \quad e_{\alpha_j} \eta = \left(\prod_{i=1}^r \lambda_i^{-A_{j i}} \right) e_{\alpha_j}, \quad 1 \leq j \leq r.$$

The determinant $d = |A_{ij}|$ has been seen in (C) (Lemma 11) to be $r+1$, if $\alpha_1, \dots, \alpha_r$ is of type A_r ; 2, if $\alpha_1, \dots, \alpha_r$ is of type B_r, C_r or E_7 ; 4, if $\alpha_1, \dots, \alpha_r$ is of type D_r ; 3, if $\alpha_1, \dots, \alpha_r$ is of type E_6 ; 1, if $\alpha_1, \dots, \alpha_r$ is of type G_2, F_4 or E_8 . Hence there are rational integers B_{ik} , $1 \leq i, k \leq r$, (which can be found

from Table 2 of [6]), such that

$$\sum_{i=1}^r A_{ji} B_{ik} = \delta_{jk} d = \sum_{i=1}^r B_{ji} A_{ik},$$

$1 \leq j, k \leq r$. Thus if $\lambda_i = \prod_{k=1}^r \rho_k^{-B_{ik}}$, (35) becomes

$$\begin{aligned} e_{\alpha_j} \eta &= \prod_{i=1}^r \left(\prod_{k=1}^r \rho_k^{-B_{ik}} \right)^{-A_{ji}} e_{\alpha_j} \\ &= \left(\prod_{i=1}^r \prod_{k=1}^r \rho_k^{A_{ji} B_{ik}} \right) e_{\alpha_j} \\ (36) \quad &= \left(\prod_{k=1}^r \prod_{i=1}^r \rho_k^{A_{ji} B_{ik}} \right) e_{\alpha_j} = \left(\prod_{k=1}^r \rho_k^{\sum_i A_{ji} B_{ik}} \right) e_{\alpha_j} \\ &= \left(\prod_{k=1}^r \rho_k^{\delta_{jk} d} \right) e_{\alpha_j} = \rho_j^d e_{\alpha_j}, \quad 1 \leq j \leq r. \end{aligned}$$

If σ is any automorphism of \mathfrak{L} such that $\mathfrak{S}\sigma = \mathfrak{S}$ and such that $\alpha_j^\sigma = \alpha_j$, we have seen in Lemma 6 that $h\sigma = h$ for all $h \in \mathfrak{S}$, and that σ is completely determined by the scalars $\mu_j \neq 0$ such that $e_{\alpha_j} \sigma = \mu_j e_{\alpha_j}$, $1 \leq j \leq r$. But if each μ_j is a d th power in \mathfrak{F} , say $\mu_j = \rho_j^d$, $1 \leq j \leq r$, and if we set

$$(37) \quad \lambda_i = \prod_{k=1}^r \rho_k^{-B_{ik}}, \quad 1 \leq i \leq r,$$

and $\eta = U(\lambda_1, 1) \cdots U(\lambda_r, 1)$, then we see by (36) that η is an invariant automorphism which coincides with σ on \mathfrak{S} and on $e_{\alpha_1}, \dots, e_{\alpha_r}$. Consequently $\sigma = \eta$ is an invariant automorphism. We have thus proved the following:

THEOREM 4. *Let \mathfrak{L} be a simple Lie algebra of classical type, \mathfrak{S} a standard Cartan subalgebra. Let $\alpha_1, \dots, \alpha_r$ be a fundamental system of roots relative to \mathfrak{S} with matrix (A_{ij}) having as determinant the natural number d . Let σ be an automorphism of \mathfrak{L} such that $\mathfrak{S}\sigma = \mathfrak{S}$ and $\alpha_i^\sigma = \alpha_i$, $1 \leq i \leq r$. Suppose further that $e_{\alpha_i} \sigma = \mu_i e_{\alpha_i}$, $1 \leq i \leq r$, where each μ_i is a d th power in \mathfrak{F} . Then σ is an invariant automorphism, and an explicit expression for σ may be found by the fact that $\sigma = \eta$, where η is constructed as in (37), with the $U(\lambda_i, i)$ as defined above.*

COROLLARY 1. *Let \mathfrak{F} be algebraically closed, and let \mathfrak{L} , \mathfrak{S} and $\alpha_1, \dots, \alpha_r$ be as in the statement of Theorem 4. Let σ be any automorphism of \mathfrak{L} such that $\mathfrak{S}\sigma = \mathfrak{S}$ and $\alpha_i^\sigma = \alpha_i$, $1 \leq i \leq r$. Then σ is an invariant automorphism. Moreover, if μ_1, \dots, μ_r are any nonzero scalars and e_{α_i} any nonzero elements of \mathfrak{L}_{α_i} , $1 \leq i \leq r$, there is an invariant automorphism η such that $h\eta = h$ for all $h \in \mathfrak{S}$ and $e_{\alpha_i} \eta = \mu_i e_{\alpha_i}$, $1 \leq i \leq r$.*

COROLLARY 2. *Let \mathfrak{L} , \mathfrak{S} , $\alpha_1, \dots, \alpha_r$ be as in the statement of Theorem 4, and suppose that \mathfrak{L} (or $\alpha_1, \dots, \alpha_r$) is of one of the types G_2, F_4, E_8 . Let σ be any*

automorphism of \mathfrak{L} such that $\mathfrak{H}\sigma = \mathfrak{H}$ and $\alpha_i^\sigma = \alpha_i$, $1 \leq i \leq r$. Then σ is an invariant automorphism; in fact, the condition $\alpha_i^\sigma = \alpha_i$ can be omitted from the hypotheses without changing the conclusion.

Proof. In these three cases, $d=1$; here \mathfrak{F} contains d th roots of all elements. Thus the first part of the corollary follows from Theorem 4. If $\mathfrak{H}\sigma = \mathfrak{H}$, then by Theorem 3, we have $\mathfrak{H}\sigma\tau^{-1} = \mathfrak{H}$, and the $\alpha_i^{\sigma\tau^{-1}}$ are a permutation of the α_i , for some invariant automorphism τ . By the remarks following Theorem 3, this can only be the identity permutation in these cases. Hence $\sigma\tau^{-1}$ is an invariant automorphism by the first part of Corollary 2, and therefore σ is an invariant automorphism.

V. The algebraically closed case. Let \mathfrak{L} , \mathfrak{H} , and $\alpha_1, \dots, \alpha_r$ be as above, and let the base field \mathfrak{F} be algebraically closed. Let σ be an automorphism of \mathfrak{L} . Then by Theorem 1 there is an invariant automorphism τ such that $\mathfrak{H}\sigma\tau^{-1} = \mathfrak{H}$. Then $\sigma\tau^{-1}$ maps $\alpha_1, \dots, \alpha_r$ onto another fundamental system with respect to \mathfrak{H} , and by Theorem 3 there is an invariant automorphism ω of \mathfrak{L} such that $\mathfrak{H}\sigma\tau^{-1}\omega^{-1} = \mathfrak{H}$ and such that $\sigma\tau^{-1}\omega^{-1}$ permutes the system $\alpha_1, \dots, \alpha_r$. Let ϕ be that permutation of the indices $1, \dots, r$ such that $\alpha_i\sigma\tau^{-1}\omega^{-1} = \alpha_{i\phi}$, $1 \leq i \leq r$. Let $0 \neq e_{\alpha_i} \in \mathfrak{L}_{\alpha_i}$, $1 \leq i \leq r$, and let $e_{\alpha_i}\sigma\tau^{-1}\omega^{-1} = \mu_i e_{\alpha_{i\phi}}$, $1 \leq i \leq r$. Then by Corollary 1, there is an invariant automorphism η such that $\mathfrak{H}\sigma\tau^{-1}\omega^{-1}\eta^{-1} = \mathfrak{H}$ and such that $e_{\alpha_{i\phi}}\eta = \mu_i e_{\alpha_{i\phi}}$, $1 \leq i \leq r$, or $\mu_i e_{\alpha_{i\phi}}\eta^{-1} = e_{\alpha_{i\phi}}$; that is, $e_{\alpha_i}\sigma\tau^{-1}\omega^{-1}\eta^{-1} = e_{\alpha_{i\phi}}$, $1 \leq i \leq r$. Thus $\sigma\tau^{-1}\omega^{-1}\eta^{-1}$ is an automorphism of \mathfrak{L} mapping \mathfrak{H} onto \mathfrak{H} and e_{α_i} onto $e_{\alpha_{i\phi}}$, $1 \leq i \leq r$. If ϕ is the identity permutation, we see by Lemma 6 that $\sigma\tau^{-1}\omega^{-1}\eta^{-1}$ is the identity automorphism, and, by III, this is necessarily the case unless \mathfrak{L} is of type A_r ($r \geq 2$), D_r ($r \geq 4$), or E_6 . Thus $\sigma = \eta\omega\tau$ is an invariant automorphism except in these cases, and we have proved:

THEOREM 5. Let \mathfrak{F} be algebraically closed, and suppose that \mathfrak{L} is not of type A_r ($r \geq 2$), of type D_r ($r \geq 4$) or of type E_6 . Then every automorphism of \mathfrak{L} is an invariant automorphism.

In general, let \mathfrak{I} be the group of invariant automorphisms, and consider the automorphism $\sigma\tau^{-1}\omega^{-1}\eta^{-1}$ mapping \mathfrak{H} onto \mathfrak{H} and e_{α_i} onto $e_{\alpha_{i\phi}}$, $1 \leq i \leq r$. Such an automorphism exists by the proof of Theorem 9 of [13] and is unique by Lemma 6. Let us denote this automorphism by $\zeta(\phi)$. Then $\sigma = \zeta(\phi)\eta\omega\tau \in \zeta(\phi)\mathfrak{I}$. In case \mathfrak{L} is of type A_r ($r \geq 2$), D_r ($r \geq 5$), or E_6 , there are only two possible values for ϕ , consequently for $\zeta(\phi)$; hence \mathfrak{I} is a subgroup of index at most two in the full automorphism group \mathfrak{A} . In case \mathfrak{L} is of type D_4 , \mathfrak{I} is a subgroup of \mathfrak{A} of index at most 6. In any case, \mathfrak{I} is readily seen to be a normal subgroup of \mathfrak{A} . With the exception of the case where \mathfrak{L} is of type E_6 , it is known that the indices are precisely 2 and 6, respectively, and in case \mathfrak{L} is of type E_6 , the results in the complex case indicate that the index is 2. The author proposes to elaborate on the results for the individual simple

algebras in a sequel to this paper. Our results for the algebras not covered by Theorem 5 are as follows:

THEOREM 6. *If \mathfrak{F} is algebraically closed and if \mathfrak{L} is of type A_r ($r \geq 2$) or of type D_r ($r \geq 5$), then the invariant automorphisms form a subgroup of index 2 in the full automorphism group. If \mathfrak{L} is of type D_4 , the invariant automorphisms form a normal subgroup of index 6 in the full automorphism group, and the quotient group is the symmetric group on three letters. If \mathfrak{L} is of type E_6 , the indicated index is 2.*

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